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Discrete-time Indefinite LQ Control with State and Control Dependent Noises

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Abstract. This paper deals with the discrete-time stochastic LQ problem involving state and control dependent noises, whereas the weighting matrices in the cost function are allowed to be indefinite. In this general setting, it is shown that the well-posedness and the attainability of the LQ problem are equivalent. Moreover, a generalized difference Riccati equation is introduced and it is proved that its solvability is necessary and sufficient for the existence of an optimal control which can be either of state feedback or open-loop form. Furthermore, the set of *all* optimal controls is identified in terms of the solution to the proposed difference Riccati equation.

Key words: Indefinite stochastic LQ control, Discrete time, Multiplicative noise, Generalized difference Riccati equation, Linear matrix inequality

1. Introduction

Optimal control has found profound applications in a wide range of practical problems. For the systems whose components are perturbed by a Gaussian noise, the most popular problem is known as the linear–quadratic–Gaussian (LQG) problem [5] and its theory has been well established. However, many real systems are subject to stochastic perturbations not necessarily of the Gaussian type. In this paper we are concerned with a general stochastic optimal control of discrete-time linear systems in which the parameters are subject to (non-Gaussian) noises not only additively but also multiplicatively in both the state and the control. The cost function (payoff function) is the expectation of an *indefinite* quadratic form in the state and the control.

Since it has been introduced by Kalman [9], the classical Riccati equation constitutes the computationally most efficient and the theoretically most important ingredient in the linear–quadratic (LQ) control design methodology. It is well-known that for the definite LQ problem the optimal control is always unique and has a feedback form with deterministic gain given by the solution to the Riccati equation. For the discrete-time LQ control, there have been some works in literature for problems with control and/or state dependent noises. One early work [11] deals with a special case, whose system dynamics are described by a difference equation in which both the system matrix and control matrix are multiplied by white, possibly correlated, scalar random sequences. Recently, in [6], the optimal control law is derived for the systems with only control dependent noises. However, in both papers it is assumed that the state weight is nonnegative and the control weight is positive definite in the performance index.

Work on continuous-time stochastic LQ control with indefinite weighting matrices can be found in a series of articles (see, e.g., [1, 2, 7], and Chapter 6 of [15]). One of the interesting applications of this indefinite LQ control is in mathematics finance [10, 16]. For discrete-time LQ problem, the control weighting matrix is not required to be positive definite even in the deterministic case. However, the control weighting matrix can be even more negative when uncertainty factors are involved in the system as will be demonstrated in this paper.

In this paper we introduce a *generalized difference Riccati equation* (GDRE) involving a matrix pseudo inverse. We show that in general the GDRE solution leads to a family of optimal controls for the indefinite stochastic LQ problem under consideration. Meanwhile we introduce a *linear matrix inequality* (LMI) condition, and prove that the feasibility of this LMI, the solvability of the GDRE, the well-posedness of the LQ problem, and the attainability of the LQ problem are all equivalent. It should be noted that a special case involving matrix (normal) inverse has been partially studied in [12]. In contrast in this paper we provide a complete solution to the problem.

The remainder of the paper is organized as follows. Section 2 formulates the indefinite stochastic LQ problem and introduces the generalized difference Riccati equation. In Section 3 the optimal state feedback control is studied using the maximum principle approach. Section 4 provides a complete solution to the LQ problem via the dynamic programming approach combined with some algebraic results. The equivalence between the well-posedness, the attainability of the LQ problem, the feasibility of the LMIs and the solvability of the GDRE is established. Section 5 shows that in general the form of an optimal control can be more complex than that of a purely static state feedback control. A characterization of the structure of the optimal controls is given. Section 6 presents an extension of the results when the noises in the system are correlated. A numerical example is reported in Section 7. Finally, Section 8 concludes the paper.

2. Problem formulation and preliminaries

We make use of the following basic notation in this paper: $\mathbf{R}^n$ is the real n-dimensional Euclidean space; $\mathbf{R}^{m\times n}$ the set of all $m \times n$ matrices; M' the transpose of a matrix M and $M^\dagger$ its Moore-Penrose pseudo inverse; and $\mathbf{Tr}(M)$ the trace of a square matrix M. Moreover, $M > 0$ (resp. $M \geqslant 0$) means that $M = M'$ and M is positive (resp. positive semi-) definite. Finally, $\mathbf{E}[x]$ represents the expectation of a random variable x.

Consider the discrete-time stochastic model

$$\begin{cases} x_{i+1} = (A_i + w_i^x C_i)x_i + (B_i + w_i^u D_i)u_i + w_i, \ i = 0, \dots, N-1, \\ x_0 \in \mathbf{R}^n \text{ is a given initial random variable }, \end{cases} \tag{1}$$

where the initial state x_0, the noises $(w_i^x, w_i^u, w_i, i = 0, \dots, N-1)$, and the control sequence $u_0, \dots, u_{N-1}, u_i \in \mathbf{R}^n$, are defined on a given probability space $(\Omega, \mathcal{B}, \mathbf{P})$. All the coefficients A_i, B_i, C_i and D_i are assumed to be deterministic matrices with appropriate dimensions determined from the context. Without loss of generality, the noises w_i^x and w_i^u in the state and control terms are assumed to be scalar random variables.

We assume that the initial condition x_0 is independent of the noises w_i^x, w_i^u, w_i, $i = 0, \dots, N-1$, and all the noises have zero means:

$$\mathbf{E}[w_i^x] = \mathbf{E}[w_i^u] = 0, \quad \mathbf{E}[w_i] = 0,$$

with the following variances/covariances

$$\mathbf{E}[(w_i^x)^2] = \mathbf{E}[(w_i^u)^2] = 1, \quad \mathbf{E}[w_i w_i'] = V_i,$$

$$\mathbf{E}[w_i^x w_i^u] = \rho_i^{xu}, \quad \mathbf{E}[w_i w_i^x] = 0, \quad \mathbf{E}[w_i w_i^u] = 0.$$

The cost function associated with the system is

$$J(x_0, u_0, \dots, u_{N-1}) = \mathbf{E}\left[\sum_{i=0}^{N-1}(x_i' Q_i x_i + u_i' R_i u_i) + x_N' Q_N x_N\right], \tag{2}$$

where $Q_0, \dots, Q_N$ and $R_0, \dots, R_{N-1}$ are symmetric matrices with appropriate dimensions.

In general, a control is defined as a sequence $(u_0, \dots, u_{N-1})$ of random variables defined on the probability space $(\Omega, \mathcal{B}, \mathbf{P})$. The *admissible control set* $\mathcal{U}_{ad}$ is the set of all such controls. The LQ problem under consideration is to find a control that minimizes $J(x_0, u_0, \dots, u_{N-1})$ over the admissible control set. We also define

$$V(x_0) \triangleq \inf_{u_0, \cdots, u_{N-1}} J(x_0, u_0, \dots, u_{N-1}). \tag{3}$$

Note that the above optimization problem may be ill-posed since the weighting matrices $Q_0, \dots, Q_N$, $R_0, \dots, R_{N-1}$ are possibly indefinite. Therefore, we have the following definition.

DEFINITION 2.1. The LQ problem (1)–(3) is called well-posed if

$$V(x_0) = \inf_{u_0, \cdots, u_{N-1}} J(x_0, u_0, \dots, u_{N-1}) > -\infty,$$

for any random variable x_0 which is independent of the noises

$$w_i^x, w_i^u, w_i, \quad i = 0, \dots, N-1.$$

The LQ problem is called attainable if there exist an admissible control $(u_0^*, \dots, u_{N-1}^*)$ such that $V(x_0) = J(x_0, u_0^*, \dots, u_{N-1}^*)$ for any random variable x_0. In this case $(u_0^*, \dots, u_{N-1}^*)$ is called an optimal control.

It is interesting to note that if the LQ problem is attainable, then we will show that there must be an optimal control that is *adaptive*, namely, a control sequence $u_0, \dots, u_{N-1}$ where each u_i is generated by the history of the state $x_0, \dots, x_i$:

$$u_i = \phi_i(x_0, \dots, x_i), \quad i = 0, \dots, N-1. \tag{4}$$

For later use, we recall the pseudo-inverse of a matrix. Let a matrix $M \in \mathbf{R}^{m\times n}$ be given. Then there exists a unique matrix $M^\dagger \in \mathbf{R}^{n\times m}$, which is called the Moore-Penrose pseudo inverse [13] of M, such that

$$\begin{cases} MM^\dagger M = M, M^\dagger MM^\dagger = M^\dagger, \\ (MM^\dagger)' = MM^\dagger, (M^\dagger M)' = M^\dagger M. \end{cases} \tag{5}$$

LEMMA 2.1. *Let a symmetric matrix S be given. Then*

(i) $S^\dagger = S^{\dagger'}$;

(ii) $S \geqslant 0$ *if and only if* $S^\dagger \geqslant 0$;

(iii) $SS^\dagger = S^\dagger S$.

Now we introduce the generalized difference Riccati equation.

DEFINITION 2.2. The following constrained difference equation

$$\begin{cases} P_i = A_i' P_{i+1} A_i - H_i' G_i^\dagger H_i + Q_i + C_i' P_{i+1} C_i, \\ P_N = Q_N, \\ G_i G_i^\dagger H_i - H_i = 0, \text{ and } G_i \geqslant 0 \text{ for } i = N-1, \dots, 0, \end{cases} \tag{6}$$

where

$$\begin{cases} H_i = B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i, \\ G_i = R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i, \quad \text{for } i = N-1, \dots, 0, \end{cases} \tag{7}$$

is called a generalized difference Riccati equation (GDRE).

3. State feedback control

In this part, we establish a link between the existence of an optimal control in state feedback form with deterministic gains and the solvability of the GDRE. The idea is to transform the stochastic LQ problem into an equivalent deterministic optimization problem subject to a matrix difference equation constraint involving only the covariance matrices of the state and the gain matrices of the feedback control. Then we apply the deterministic maximum principle [4]. This approach also gives a nice interpretation to the solution of the GDRE, which is nothing but the matrix Lagrangian multiplier (see the proof of Theorem 3.1 below).

LEMMA 3.1. *Let matrices L, M and N be given with appropriate sizes. Then the following matrix equation*

$$LXM = N, \tag{8}$$

has a solution X if and only if

$$LL^\dagger NMM^\dagger = N. \tag{9}$$

Moreover, any solution to (8) is represented by

$$X = L^\dagger NM^\dagger + Y - L^\dagger LYMM^\dagger, \tag{10}$$

where Y is a matrix with an appropriate size.

THEOREM 3.1. *If the LQ problem (1)-(3) is attainable by a feedback control law*

$$u_i = K_i x_i, \quad \text{for } i = 1, \dots, N-1, \tag{11}$$

where $K_0, \dots, K_N$ are constant deterministic matrices, then the GDRE (6) has a solution.

Proof. The first step is to formulate the LQ problem in terms of the state covariance matrices $X_i = E[x_i x_i']$ and the gain matrices K_i. By a simple calculation it can be seen that the following deterministic optimal control problem is equivalent to the original problem (1)–(3) with a feedback control of the form (11):

$$\min_{K_0,\dots,K_{N-1}\in\mathbf{R}^{m\times n}} \sum_{i=0}^{N-1} \mathbf{Tr}[(Q_i + K_i' R_i K_i)X_i] + \mathbf{Tr}(Q_N X_N),$$

subject to

$$\begin{cases} X_{i+1} = A_i X_i A_i' + C_i X_i C_i' + B_i K_i X_i K_i' B_i' + D_i K_i X_i K_i' D_i' \\ \qquad\qquad + A_i X_i K_i' B_i' + B_i K_i X_i A_i' + \rho_i^{xu} D_i K_i X_i C_i' + \rho_i^{xu} C_i X_i K_i' D_i' + V_i, \\ X_0 \text{ is a given symmetric matrix.} \end{cases} \tag{12}$$

In the above problem the matrices $K_0, \dots, K_{N-1}$ are viewed as the control to be determined. Hence, as in [4] we can apply the matrix Lagrangian multiplier method to solve the above problem. The Lagrangian function is formed as

$$\mathcal{L} = \sum_{i=0}^{N-1} \mathcal{H}_i + \mathbf{Tr}(Q_N X_N),$$

where

$$\begin{aligned} \mathcal{H}_i \triangleq{} & \mathbf{Tr}[(Q_i + K_i' R_i K_i)X_i] \\ & + \mathbf{Tr}[P_{i+1}\Big(A_i X_i A_i' + C_i X_i C_i' + B_i K_i X_i K_i' B_i' + D_i K_i X_i K_i' D_i' \\ & + A_i X_i K_i' B_i' + B_i K_i X_i A_i' + \rho_i^{xu} D_i K_i X_i C_i' + \rho_i^{xu} C_i X_i K_i' D_i' + V_i - X_{i+1}\Big)], \end{aligned}$$

and the matrices $P_0, \dots, P_{N-1}$ are the Lagrangian multipliers. The first-order necessary conditions for optimality [4] are found to be

$$\begin{cases} \dfrac{\partial \mathcal{H}_i}{\partial K_i} = 0,\ P_i = \dfrac{\partial \mathcal{H}_i}{\partial X_i}, \quad \text{for } i = 1, \dots, N-1, \\ P_N = Q_N. \end{cases}$$

The calculation of the above derivatives leads to the following equations

$$(R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i) K_i + B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i = 0, \tag{13}$$

$$\begin{aligned} P_i &= Q_i + A_i' P_{i+1} A_i + C_i' P_{i+1} C_i + K_i'(R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i) K_i \\ &\quad + (A_i' P_{i+1} B_i + \rho_i^{xu} C_i' P_{i+1} D_i) K_i + K_i'(B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i), \\ P_N &= Q_N. \end{aligned} \tag{14}$$

Now by using Lemma 3.1 we can see that the existence of a solution $K_0, \dots, K_{N-1}$ to Eq. (13) is equivalent to $G_i^\dagger G_i H_i - H_i = 0$, where

$$\begin{aligned} G_i &= R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i \\ H_i &= B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i, \end{aligned}$$

and the general solution is given by the following gains

$$K_i = -G_i^\dagger H_i + Y_i - G_i^\dagger G_i Y_i, \quad Y_i \in \mathbf{R}^{m\times n}, \ \text{for } i = 1, \dots, N-1.$$

Putting the above gains into (14) we obtain

$$\begin{cases} P_i = A_i' P_{i+1} A_i - H_i' G^\dagger H_i + C_i' P_{i+1} C_i + Q_i, \\ P_N = Q_N, \\ G_i^\dagger G_i H_i - H_i = 0, \\ H_i = B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i, \\ G_i = R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i. \end{cases} \tag{15}$$

Notice that the above equation is exactly the GDRE without any positiveness constraint. The proof will be complete if we show that $G_i \geqslant 0$, $i = 0, \dots, N-1$.

Let us suppose there is G_l with an associated negative eigenvalue λ. Denote the unitary eigenvector respect with to λ as v_λ (i.e., $v_\lambda' v_\lambda = 1$ and $G_l v_\lambda = \lambda v_\lambda$). Let $\delta \neq 0$ be an arbitrary scalar and construct a control sequence as follows

$$\tilde{u}_i = \begin{cases} -G_i^\dagger H_i x_i, & i \neq l, \\ \delta |\lambda|^{-\frac{1}{2}} v_\lambda - G_i^\dagger H_i x_i, & i = l. \end{cases}$$

The associated cost is

$$\begin{aligned} & J(x_0, \tilde{u}_0, \dots, \tilde{u}_{N-1}) \\ & = \mathbf{E}\sum_{i=0}^{N-1}[(\tilde{u}_i + G_i^{\dagger}H_i x_i)' G_i(\tilde{u}_i + G_i^{\dagger}H_i x_i)] + \sum_{i=0}^{N-1}\mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0] \\ & = \left(\frac{\delta}{|\lambda|^{\frac{1}{2}}}v_\lambda\right)' G_l \left(\frac{\delta}{|\lambda|^{\frac{1}{2}}}v_\lambda\right) + \sum_{i=0}^{N-1}\mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0] \\ & = -\delta^2 + \sum_{i=0}^{N-1}\mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0]. \end{aligned}$$

Obviously, as $\delta \to \infty$, $J(x_0, \tilde{u}_0, \dots, \tilde{u}_{N-1}) \to -\infty$ which contradicts the assumption of the theorem. □

4. Well-posedness and attainability

In this section, the connection between the well-posedness and attainability of the LQ problem is examined. It turns out, rather surprisingly, that when the optimal cost value is finite the LQ problem is always achievable by an optimal control. To this end, we first establish the link between the well-posedness and some LMI condition, and then prove that the LMI condition is equivalent to the the solvability of the GDRE as well as to the attainability of the LQ problem.

4.1. AN LMI CONDITION

The following result formulates the well-posedness as the feasibility of some LMI involving unknown symmetric matrices. In fact, here we only show that the proposed LMI condition is sufficient for the well-posedness. The necessity will be shown later.

THEOREM 4.1. *The LQ problem (1)-(3) is well-posed if there exist symmetric matrices $P_0, \dots, P_N$ satisfying the following LMI condition*

$$\left[\begin{array}{c|c} A_i' P_{i+1} A_i - P_i + C_i' P_{i+1} C_i + Q_i & A_i' P_{i+1} B_i + \rho_i^{xu} C_i' P_{i+1} D_i \\ \hline B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i & R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i \end{array}\right] \geqslant 0, \tag{16}$$

for $i = 0, \dots, N-1$, and $P_N \leqslant Q_N$.

Proof. Let $P_1, \dots, P_N$ satisfy (16). Then by adding the following trivial equality

$$\sum_{i=0}^{N-1}(x_{i+1}' P_{i+1} x_{i+1} - x_i' P_i x_i) = \mathbf{E}[x_N' P_N x_N - x_0' P_0 x_0] \tag{17}$$

to the cost function

$$J(x_0, u_0, \dots, u_{N-1}) = \mathbf{E}[\sum_{i=0}^{N-1}(x_i' Q_i x_i + u_i' R_i u_i) + x_N' Q_N x_N],$$

and using the system Eq. (1), we can rewrite the cost function as follows

$$\begin{aligned}
&J(x_0, u_0, \dots, u_{N-1}) \\
&\quad = \mathbf{E}\sum_{i=0}^{N-1}(x_i' Q_i x_i + u_i' R_i u_i + x_{i+1}' P_{i+1} x_{i+1} - x_i' P_i x_i)] + \mathbf{E}[x_N'(Q_N - P_N)x_N + x_0' P_0 x_0] \\
&\quad = \mathbf{E}\sum_{i=0}^{N-1}[x_i'(Q_i - P_i + A_i' P_{i+1} A_i + C_i' P_{i+1} C_i)x_i + 2x_i'(A_i' P_{i+1} B_i + \rho_i^{xu} C_i' P_{i+1} D_i)u_i \\
&\qquad + u_i'(R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i)u_i] + \mathbf{E}\sum_{i=0}^{N-1} w_i' P_{i+1} w_i \\
&\qquad + \mathbf{E}[x_N'(Q_N - P_N)x_N + x_0' P_0 x_0] \\
&\quad = \mathbf{E}\sum_{i=0}^{N-1}\begin{bmatrix} x_i \\ u_i \end{bmatrix}'\left[\begin{array}{c|c} A_i' P_{i+1} A_i + C_i' P_{i+1} C_i + Q_i - P_i & A_i' P_{i+1} B_i + \rho_i^{xu} C_i' P_{i+1} D_i \\ \hline B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i & R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i \end{array}\right]\begin{bmatrix} x_i \\ u_i \end{bmatrix} \\
&\qquad + \sum_{i=0}^{N-1}\mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_N'(Q_N - P_N)x_N + x_0' P_0 x_0].
\end{aligned}$$

From the above equality we can easily deduce that the cost function is bounded from below by $\mathbf{E}[x_0' P_0 x_0] + \sum_{i=0}^{N-1}\mathbf{Tr}(V_i P_{i+1})$ and hence the LQ problem (1)-(3) is well-posed. □

REMARK 4.1. We have shown in the proof of Theorem 4.1 that any symmetric matrices $P_0, \dots, P_N$ satisfying condition (16) provide a lower bound, $\mathbf{E}(x_0' P_0 x_0) + \sum_{i=0}^{N-1}\mathbf{Tr}(V_i P_{i+1})$, for the optimal cost value. In the next subsection, we will see that this lower bound becomes the exact optimal cost value when $P_0, \dots, P_N$ solve the GDRE.

4.2. EQUIVALENCE BETWEEN WELL-POSEDNESS AND ATTAINABILITY

In the preceding subsection we have shown that if the LMI condition (16) is satisfied then the well-posedness of the LQ problem holds. In this subsection we not only prove the reverse implication but also show that the well-posedness, the attainability, the LMI condition, and the solvability of the GDRE are equivalent to each other.

The following lemmas are useful in our analysis.

LEMMA 4.1 (Extended Schur's lemma [3]). *Let be given matrices $F = F'$, H and $G = G'$ with appropriate sizes. Then the following conditions are equivalent:*

(i) $F - HG^{\dagger}H' \geqslant 0$, $G \geqslant 0$, *and* $H(I - GG^{\dagger}) = 0$.

(ii) $\begin{bmatrix} F & H \\ H' & G \end{bmatrix} \geqslant 0$.

(iii) $\begin{bmatrix} G & H' \\ H & F \end{bmatrix} \geqslant 0$.

LEMMA 4.2. *Let be given matrices $G = G'$ and H with appropriate sizes. Then the following conditions are equivalent*

(i) $H(I - GG^{\dagger}) = 0$.

(ii) $Ker(G) \subseteq Ker(H)$.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial. On the other hand, the reverse implication can be shown by using the singular value decomposition of G (see [8]):

$$G = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V',$$

where Σ is a nonsingular diagonal matrix and V a matrix such that $VV' = I$. Moreover, V is decomposed as $V = [V_1 \;\; V_2]$ where the columns of the matrix V_2 form a basis of $\ker G$. Now, one can deduce straightforwardly that G^{+} is given by

$$G^{+} = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} V'.$$

A simple calculation yields $H(I - GG^{\dagger}) = HV_2V_2'$. Since $GV_2 = 0$ we have $HV_2 = 0$ and the proof is complete. □

LEMMA 4.3. *Let be given matrices $F = F'$, H and $G = G'$ with appropriate sizes. Consider the following quadratic form*

$$q(x, u) = \mathbf{E}[x'Fx + 2x'Hu + u'Gu],$$

where x and u are random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$. Then the following conditions are equivalent:

(i) $\inf_u q(x, u) > -\infty$ *for any random variable x.*

(ii) *There exists a symmetric matrix $S = S'$ such that* $\inf_u q(x, u) = \mathbf{E}[x'Sx]$, *for any random variable x.*

(iii) $G \geqslant 0$ *and* $H(I - GG^{\dagger}) = 0$.

(iv) $G \geqslant 0$ *and* $Ker(G) \subseteq Ker(H)$.

(v) *There exists a symmetric matrix $T = T'$ such that* $\begin{bmatrix} F - T & H \\ H' & G \end{bmatrix} \geqslant 0$.

Moreover, if any of the above condition holds, then (ii) is satisfied by $S = F - HG^{\dagger}H'$*. In addition,* $S \geqslant T$ *for any* T *satisfying (v). Finally, for any random variable* x*, the random variable* $u^* = -G^{\dagger}H'x$ *is optimal with the following optimal value*

$$q(x, u^*) = \mathbf{E}[x'(F - HG^{\dagger}H')x].$$

Proof. The required equivalences are proved by the following loop:

$$(i) \Rightarrow (iv) \Leftrightarrow (iii) \Rightarrow (ii) \Rightarrow (v) \Rightarrow (i).$$

First note that the equivalence $(iv) \Leftrightarrow (iii)$ is nothing than the result given by Lemma 4.2.
$(i) \Rightarrow (iv)$: Assume that there exists v such that $v'Gv < 0$. Then for an arbitrary scalar $\alpha > 0$ we have $\lim_{\alpha \to +\infty} q(x, \alpha v) = -\infty$. By assumption this leads to contradiction. Hence G must be positive.

Suppose now that $Ker(G) \not\subseteq Ker(H)$. In other words there exists u such that $Gu = 0$ and $Hu \neq 0$. Take any arbitrary scalar $\alpha > 0$ then it is immediate that $\lim_{\alpha \to +\infty} q(Hu, -\alpha u) = -\infty$ which contradicts (i).
$(iii) \Rightarrow (ii)$: A simple calculation gives

$$q(x, u) = \mathbf{E}[x'(F - HG^{\dagger}H')x + (u' + x'HG^{\dagger})G(u + G^{\dagger}H'x)].$$

Define $S = F - HG^{\dagger}H'$, then it is easily seen that $\inf_u q(x, u) = \mathbf{E}[x'Sx]$ for any random variable x.
$(ii) \Rightarrow (v)$: We have for any random variables x, u:

$$q(x, u) = \mathbf{E}[x'Fx + 2x'Hu + u'Gu] \geqslant \mathbf{E}[x'Sx],$$

or equivalently

$$\mathbf{E}\begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} F - S & H \\ H' & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \geqslant 0.$$

Thus (v) holds with $T = S$.
$(v) \Rightarrow (i)$: trivial.

The rest of the proof is straightforward. □

The following provides a connection between the well-posedness of the LQ problem and the solvability of the GDRE.

THEOREM 4.2. *The LQ problem (1)-(3) is well-posed if and only if there exist symmetric matrices* $P_0, \ldots, P_N$ *satisfying the GDRE (6). Furthermore the optimal cost is given by*

$$\inf_{u_0, \ldots, u_{N-1}} J(x_0, u_0, \ldots, u_{N-1}) = \sum_{i=0}^{N-1} \mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0]. \tag{18}$$

Proof. We prove the first assertion by induction. To this end consider the cost-to-go from l to N

$$V^l(x_l) = \inf_{u_l,\dots,u_{N-1}} \mathbf{E}\left[\sum_{i=l}^{N-1}(x_i' Q_i x_i + u_i' R_i u_i) + x_N' Q_N x_N\right]. \tag{19}$$

Note that by the stochastic optimality principle when $V^{l_1}(x_{l_1})$ is finite then $V^{l_2}(x_{l_2})$ is also finite for any $l_1 \leqslant l_2$. This fact will be used at each step of the induction: since the LQ problem is assumed to be well-posed at the initial (zero) time, the cost-to-go $V^l(x_l)$ is finite at any stage $0 \leqslant l \leqslant N-1$.

Now let us start with $l = N-1$ and define $P_N = Q_N$. Then by using the system equation (1) and (19) we have

$$\begin{aligned}
&V^{N-1}(x_{N-1}) - \mathbf{Tr}(V_{N-1}P_N)\\
&\quad = \inf_{u_{N-1}} \mathbf{E}[x_{N-1}' Q_{N-1} x_{N-1} + u_{N-1}' R_{N-1} u_{N-1} + x_N' Q_N x_N] - \mathbf{Tr}(V_{N-1}P_N)\\
&\quad = \inf_{u_{N-1}} E[x_{N-1}'(Q_{N-1} + A_{N-1}' Q_N A_{N-1} + C_{N-1}' Q_N C_{N-1}) x_{N-1}\\
&\qquad + 2x_{N-1}'(A_{N-1}' Q_N B_{N-1} + \rho_{N-1}^{xu} C_{N-1}' Q_N D_{N-1}) u_{N-1}\\
&\qquad + u_{N-1}'(R_N + B_{N-1}' Q_N B_{N-1} + D_{N-1}' Q_N D_{N-1}) u_{N-1}].
\end{aligned}$$

Since $V^{N-1}(x_{N-1})$ is finite, using Lemma 4.3 we can guarantees the existence of a symmetric matrix P_{N-1} such that

$$V^{N-1}(x_{N-1}) - \mathbf{Tr}(V_{N-1}P_N) = \mathbf{E}[x_{N-1}' P_{N-1} x_{N-1}].$$

Also by Lemma 4.3 we have the following conditions which are nothing else than the GDRE (6) for $i = N-1$:

$$\begin{aligned}
P_{N-1} &= A_{N-1}' P_N A_{N-1} - H_{N-1}' G_{N-1}^{\dagger} H_{N-1} + C_{N-1}' P_{i+1} C_{N-1} + Q_{N-1},\\
G_{N-1} &= R_{N-1} + B_{N-1}' P_N B_{N-1} + D_{N-1}' P_N D_{N-1} \geqslant 0,\\
H_{N-1} &= B_{N-1}' P_N A_{N-1} + \rho_{N-1}^{xu} D_{N-1}' P_N C_{N-1}.
\end{aligned}$$

Now suppose that we have found a sequence of symmetric matrices $P_l, \dots, P_{N-1}$ which solve the GDRE (6) for $i = N-1, \dots, l$, and satisfy

$$V^l(x_l) - \sum_{i=l}^{N-1} \mathbf{Tr}(V_i P_{i+1}) = \mathbf{E}(x_l' P_l x_l).$$

Then by the stochastic optimality principle the following is derived:

$$\begin{aligned}
&V^{l-1}(x_{l-1}) - \sum_{i=l-1}^{N-1} \mathbf{Tr}(V_i P_{i+1}) \\
&\quad = \inf_{u_{l-1}} E[x_{l-1}' Q_{l-1} x_{l-1} + u_{l-1}' R_{l-1} u_{l-1} + V^l(x_l)] - \sum_{i=l-1}^{N-1} \mathbf{Tr}(V_i P_{i+1}) \\
&\quad = \inf_{u_{l-1}} E(x_{l-1}' Q_{l-1} x_{l-1} + u_{l-1}' R_{l-1} u_{l-1} + x_l' P_l x_l) - \mathbf{Tr}(V_{l-1} P_l) \\
&\quad = \inf_{u_{l-1}} E[x_{l-1}'(Q_{l-1} + A_{l-1}' P_l A_{l-1} + C_{l-1}' P_l C_{l-1}) x_{l-1} \\
&\qquad + 2x_{l-1}'(A_{l-1}' P_l B_{l-1} + \rho_{l-1}^{xu} C_{l-1}' Q_l D_{l-1}) u_{l-1} \\
&\qquad + u_{l-1}'(R_l + B_{l-1}' P_l B_{l-1} + D_{l-1}' P_l D_{l-1}) u_{l-1}].
\end{aligned}$$

As in the preceding we can see that Lemma 4.3 provides the following necessary and sufficient conditions for the finiteness of $V^{l-1}(x_{l-1})$:

$$\begin{cases}
P_{l-1} = A_{l-1}' P_l A_{l-1} - H_{l-1}' G_{l-1}^{\dagger} H_{l-1} + Q_{l-1} + C_{l-1}' P_l C_{l-1}, \\
G_{l-1} = R_{l-1} + B_{l-1}' P_l B_{l-1} + D_{l-1}' P_l D_{l-1} \geqslant 0, \\
G_{l-1} G_{l-1}^{\dagger} H_{l-1} - H_{l-1} = 0, \\
H_{l-1} = B_{l-1}' P_l A_{l-1} + \rho_{l-1}^{xu} D_{l-1}' P_l C_{l-1}.
\end{cases}$$

In addition,

$$V^{l-1}(x_{l-1}) = \sum_{i=l-1}^{N-1} \mathbf{Tr}(V_i P_{i+1}) + x_{l-1}' P_{l-1} x_{l-1}.$$

These prove the first assertion. Next, noticing that by Lemma 4.1 the solution to the GDRE satisfies also the LMI condition (16) which by Theorem 4.1 implies the well-posedness of the LQ problem. □

The main result of this section is given by the following.

THEOREM 4.3. *The following are equivalent*

(i) *The LQ problem is well-posed.*

(ii) *The LQ problem is attainable.*

(iii) *The LMI condition (16) is feasible.*

(iv) *The GDRE (6) is solvable.*

Moreover, when any of the above conditions is satisfied the LQ problem is attainable by

$$\begin{aligned}
u_i = &- [R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i]^{\dagger} [B_i' P_{i+1} A_i \\
&+ \rho_i^{xu} D_i' P_{i+1} C_i] x_i, \quad i = 0, \ldots, N-1,
\end{aligned} \tag{20}$$

where $P_0, \ldots, P_N$ are solutions to the GDRE (6).

Proof. By Theorem 4.1 and Theorem 4.2 the equivalences $(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are straightforward. What remains to show is that the LQ problem is attainable by the feedback control law (20). To this end, let $P_1, \ldots, P_N$ solve the GDRE (6). Then by adding the equality

$$\sum_{i=0}^{N-1}(x'_{i+1}P_{i+1}x_{i+1} - x'_iP_ix_i) = \mathbf{E}[x'_NP_Nx_N - x'_0P_0x_0]$$

to the performance index we have

$$\begin{aligned}
&J(x_0, u_0, \ldots, u_{N-1})\\
&\quad = \mathbf{E}\sum_{i=0}^{N-1}[x'_i(Q_i - P_i + A'_iP_{i+1}A_i + C'_iP_{i+1}C_i)x_i\\
&\qquad + 2x'_i(A'_iP_{i+1}B_i + \rho_i^{xu}C'_iP_{i+1}D_i)u_i\\
&\qquad + u'_i(R_i + B'_iP_{i+1}B_i + D'_iP_{i+1}D_i)u_i] + \mathbf{E}\sum_{i=0}^{N-1}w'_iP_{i+1}w_i + \mathbf{E}[x'_0P_0x_0].
\end{aligned}$$

A completion of square yields

$$\begin{aligned}
&J(x_0, u_0, \ldots, u_{N-1})\\
&\quad = \mathbf{E}\sum_{i=0}^{N-1}[(u_i + G_i^\dagger H_ix_i)'G_i(u_i + G_i^\dagger H_ix_i)]\\
&\qquad + \sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}) + \mathbf{E}[x'_0P_0x_0],
\end{aligned}$$

which shows that the optimal value equals $\sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}) + \mathbf{E}[x'_0P_0x_0]$ and it is attainable by the feedback $u_i = -G_i^\dagger H_ix_i$ for $i = 1, \ldots, N-1$. □

REMARK 4.2. Suppose that there exist $Q_0, \ldots, Q_N$ and $R_0, \ldots, R_{N-1}$ such that the LQ problem is attainable. Then it is easily verified that for every $\tilde{Q}_0 \geqslant Q_0, \ldots, \tilde{Q}_N \geqslant Q_N$ and $\tilde{R}_0 \geqslant R_0, \ldots, \tilde{R}_{N-1} \geqslant R_{N-1}$, the LMI condition (16) is also satisfied. Therefore, any of the statements in Theorem 4.3 holds for every $\tilde{Q}_0 \geqslant Q_0, \ldots, \tilde{Q}_N \geqslant Q_N$ and $\tilde{R}_0 \geqslant R_0, \ldots, \tilde{R}_{N-1} \geqslant R_{N-1}$.

REMARK 4.3. It should be noted that the GDRE solution is also the unique solution to the following semidefinite programming (SDP) [14]

$$\min_{P_0,\ldots,P_{N-1}} -\mathbf{Tr}\sum_0^N P_i \quad \text{subject to the LMI condition (16).}$$

This fact is an immediate consequence of our previous results. However, it seems that solving the above SDP is computationally unfavorable since the GDRE solution can be found by a simple backward calculation.

5. Optimal synthesis via GDRE

It is well-known that for a definite LQ problem the optimal control is always unique and has a feedback form with deterministic gain given by the solution to the difference Riccati equation. For the indefinite case this is no longer true. In the following we provide a complete characterization of all optimal controls. Precisely, we show that any optimal control can be expressed in terms of the solution to the GDRE with two degrees of freedom. In general the optimal control involves a feedback form with random gains and an additional random term.

The following is the main result of this section.

THEOREM 5.1. *Assume that the GDRE (6) admits a solution. Then the set of all optimal controls is determined by the following (parameterized by Y_i, z_i):*

$$u_i^{(Y_i,z_i)} = -(G_i^\dagger H_i + Y_i - G_i^\dagger G_i Y_i)x_i + z_i - G_i^\dagger G_i z_i, \tag{21}$$

where $Y_i \in \mathbf{R}^{m\times n}$ and $z_i \in \mathbf{R}^m$ are arbitrary random variables defined on the probability space $(\Omega, \mathcal{B}, \mathbf{P})$. Furthermore, the optimal cost value is uniquely given by

$$\inf_{u_0,\ldots,u_{N-1}} J(x_0,u_0,\ldots,u_{N-1}) = \sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}) + \mathbf{E}[x_0'P_0x_0], \tag{22}$$

where $P_0,\ldots,P_{N-1}$ solve the GDRE.

Proof. We show first that any control law of the form (21) is optimal. Let $P_0,\ldots,P_{N-1}$ solve GDRE (6). By the same calculation used in our previous proofs the cost function can be expressed as follows

$$\begin{aligned}
&J(x_0,u_0,\ldots,u_{N-1})\\
&= \mathbf{E}\sum_{i=0}^{N-1}(x_i'Q_ix_i + u_i'R_iu_i + x_{i+1}'P_{i+1}x_{i+1} - x_i'P_ix_i) + \mathbf{E}[x_0'P_0x_0]\\
&= \mathbf{E}[\sum_{i=0}^{N-1}x_i'(Q_i - P_i + A_i'P_{i+1}A_i + C_i'P_{i+1}C_i)x_i + 2x_i'(A_i'P_{i+1}B_i + \rho_i^{xu}C_i'P_{i+1}D_i)u_i\\
&\quad + u_i'(R_i + B_i'P_{i+1}B_i + D_i'P_{i+1}D_i)u_i] + \mathbf{E}\sum_{i=0}^{N-1}w_i'P_{i+1}w_i + \mathbf{E}[x_0'P_0x_0],\\
&= \mathbf{E}\sum_{i=0}^{N-1}[x_i'H_i'G_i^\dagger H_ix_i + 2x_i'H_i'u_i + u_i'G_iu_i] + \sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}) + \mathbf{E}[x_0'P_0x_0].
\end{aligned}$$

Denote $M_i^1 = -(Y_i - G_i^\dagger G_i Y_i)$ and $M_i^2 = -(z_i - G_i^\dagger G_i z_i)$. Then we have

$$G_i M_i^1 = 0, \quad G_i M_i^2 = 0. \tag{23}$$

Thus, using the last expression of $J(x_0, u_0, \dots, u_{N-1})$ given above and the equation (23) we have the following

$$\begin{aligned} &J(x_0, u_0, \dots, u_{N-1}) \\ &= \sum_{i=0}^{N-1} \mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0] \\ &\quad + \mathbf{E} \sum_{i=0}^{N-1} (u_i + (G_i^\dagger H_i + M_i^1) x_i + M_i^2)' G_i (u_i + (G_i^\dagger H_i + M_i^1) x_i + M_i^2). \end{aligned} \tag{24}$$

Since by definition $G_i \geqslant 0$ for $i = 1, \dots, N-1$, we conclude that the control sequence $u_i = -[(G_i^\dagger H_i + M_i^1)x_i + M_i^2]$, $i = 0, \dots, N-1$, minimizes J with the optimal value given by (22).

Next, consider any control sequence $\bar{u}_0, \dots, \bar{u}_{N-1}$ which minimizes the cost function J. Then we have

$$\begin{aligned} &J(x_0, \bar{u}_0, \dots, \bar{u}_{N-1}) \\ &\quad = \mathbf{E} \sum_{i=0}^{N-1} [(\bar{u}_i + G_i^\dagger H_i x_i)' G_i (\bar{u}_i + G_i^\dagger H_i x_i)] + \sum_{i=0}^{N-1} \mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0] \\ &\quad = \sum_{i=0}^{N-1} \mathbf{Tr}(V_i P_{i+1}) + \mathbf{E}[x_0' P_0 x_0]. \end{aligned}$$

Necessarily, the above equality implies

$$\mathbf{E} \sum_{i=0}^{N-1} [(\bar{u}_i + G_i^\dagger H_i x_i)' G_i (\bar{u}_i + G_i^\dagger H_i x_i)] = 0.$$

As $G_i \geqslant 0$ for $i = 1, \dots, N-1$, we have the following equivalent condition

$$G_i(\bar{u}_i + G_i^\dagger H_i x_i) = 0, \quad i = 1, \dots, N-1.$$

Hence each $\bar{u}_i$ solves the following equation

$$G_i \bar{u}_i + G_i G_i^\dagger H_i x_i = 0.$$

Using Lemma 3.1 with $L = G_i$, $M = I$, $N = -G_i G_i^\dagger H_i x_i$, we have the following solution

$$\bar{u}_i = -G_i^\dagger H_i x_i + z_i - G_i^\dagger G_i z_i.$$

Thus $\bar{u}_i$ is represented by (21). □

In the following we present some special cases of the previous result. The first one is when the LQ problem is attainable by a unique optimal control. The second one is that the cost function has a constant value with any control sequence.

COROLLARY 5.1. *The LQ problem is uniquely solvable if and only if $G_i > 0$, for $i = 0, \dots, N-1$. Moreover, the unique optimal control is given by*

$$u_i = -G_i^{-1} H_i x_i, \quad i = 0, \dots, N-1.$$

COROLLARY 5.2. *If $G_i = 0$, $i = 0, \dots, N-1$, then any admissible control is optimal and the GDRE reduces to the following linear system:*

$$\begin{cases} P_i - A_i' P_{i+1} A_i - C_i' P_{i+1} C_i - Q_i = 0, \\ P_N = Q_N, \\ B_i' P_{i+1} A_i + \rho_i^{xu} D_i' P_{i+1} C_i = 0, \\ R_i + B_i' P_{i+1} B_i + D_i' P_{i+1} D_i = 0, \quad \textit{for } i = 0, \dots, N-1. \end{cases} \tag{25}$$

6. Extensions

So far we have assumed that $\mathbf{E}[w_i w_i^x] = \mathbf{E}[w_i w_i^u] = 0$. Here we show how to treat the case when the noises w_i^x and w_i^u are correlated with the additive noise w_i. In this situation, the optimal control requires an additional input term.

We provide in the following the optimal solution to the LQ problem in the case when $\mathbf{E}[w_i w_i^x] = \rho_i^x \neq 0$, and $\mathbf{E}[w_i w_i^u] = \rho_i^u \neq 0$.

THEOREM 6.1. *Assume that the GDRE [6] is solvable. Then the LQ problem (1)–(3) with $\rho_i^x \neq 0$, $\rho_i^u \neq 0$ is attainable by an optimal control in the following form:*

$$\bar{u}_i = -G_i^{\dagger}(H_i x_i + \psi_i), \quad i = 0, \dots, N-1, \tag{26}$$

where

$$\psi_i = D_i' P_{i+1} \rho_i^u + B_i' \phi_{i+1}, \tag{27}$$

$P_0, \dots, P_{N-1}$ solve GDRE (6) (G_i, H_i are defined as in (6)), and ϕ_i satisfies the following equation

$$\begin{cases} \phi_i = (A_i' - H_i' G_i^{\dagger} B_i')\phi_{i+1} + C_i' P_{i+1} \rho_i^x - H_i' G_i^{\dagger} D_i' \rho_i^u, \\ G_i G_i^{\dagger}(D_i' P_{i+1} \rho_i^u + B_i' \phi_{i+1}) - D_i' P_{i+1} \rho_i^u + B_i' \phi_{i+1} = 0, \\ \phi_N = 0, \end{cases} \tag{28}$$

or equivalently

$$\begin{cases} \phi_i = A_i' \phi_{i+1} + C_i' P_{i+1} \rho_i^x - H_i' G_i^{\dagger} \psi_i, \\ G_i G_i^{\dagger} \psi_i - \psi_i = 0, \\ \phi_N = 0. \end{cases} \tag{29}$$

Moreover, the optimal cost value is given by

$$\inf_{u_0,\dots,N-1} J(x_0,u_0,\dots,u_{N-1}) = \sum_{i=0}^{N-1}[\mathbf{Tr}(V_iP_{i+1}) - \psi_i' G_i^{\dagger}\psi_i] + \mathbf{E}[x_0'P_0x_0 + 2x_0'\phi_0]. \tag{30}$$

Proof. First note that

$$\mathbf{E}(x_{i+1}) = \mathbf{E}(A_ix_i + B_iu_i) \text{ and } \mathbf{E}[-x_0'\phi_0] = \mathbf{E}\sum_{i=0}^{N-1}(x_{i+1}'\ \phi_{i+1} - x_i'\phi_i).$$

By (1),(6),(17) and (29), we have

$$\begin{aligned}\mathbf{E}[x_N'P_Nx_N - x_0'P_0x_0] = &\mathbf{E}\sum_{i=0}^{N-1}[x_i'(H_i'G_i^{\dagger}H_i - Q_i)x_i + u_i'(G_i - R_i)u_i \\ &+ 2x_i'H_i'u_i + 2x_i'C_i'P_{i+1}\rho_i^x + 2u_i'D_i'P_{i+1}\rho_i^u] \\ &+ \sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}),\end{aligned}$$

and

$$\mathbf{E}[-x_0'\phi_0] = \mathbf{E}\sum_{i=0}^{N-1}[x_i'(H_i'G_i^{\dagger}\psi_i - C_i'P_{i+1}\rho_i^x) + u_i'B_i'\phi_{i+1}].$$

Hence, we have the following

$$\begin{aligned}&J(x_0,u_0,\dots,N-1) - \mathbf{E}[x_0'P_0x_0 - 2x_0'\phi_0] \\ &\quad = \mathbf{E}\sum_{i=0}^{N-1}[x_i'(H_i'G_i^{\dagger}H_i)x_i + u_i'G_iu_i + 2x_i'H_i'u_i + 2x_i'H_i'G_i^{\dagger}\psi_i + 2u_i'\psi_i] \\ &\qquad + \sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}) \\ &\quad = \mathbf{E}\sum_{i=0}^{N-1}[(u_i + G_i^{\dagger}(H_ix_i + \psi_i))'G_i(u_i + G_i^{\dagger}(H_ix_i + \psi_i)) - \psi_i'G_i^{\dagger}\psi_i] \\ &\qquad + \sum_{i=0}^{N-1}\mathbf{Tr}(V_iP_{i+1}).\end{aligned}$$

Finally, using (26) we see that

$$J(x_0,\bar{u}_0,\dots,\bar{u}_{N-1}) = \sum_{i=0}^{N-1}[\mathbf{Tr}(V_iP_{i+1}) - \psi_i'G_i^{\dagger}\psi_i] + \mathbf{E}[x_0'P_0x_0 + 2\phi_0'x_0].$$

Therefore $\bar{u}_0, \ldots, \bar{u}_{N-1}$ is an optimal control. □

REMARK 6.1. As in Theorem 5.1, the general form of the optimal control law involves two degrees of freedom Y_i, z_i. The set of optimal controls is given by

$$\bar{u}_i^{(Y_i,z_i)} = -(G_i^\dagger H_i + Y_i - G_i^\dagger G_i Y_i)x_i - G_i^\dagger \psi_i + z_i - G_i^\dagger G_i z_i, \tag{31}$$

where $Y_i \in \mathbf{R}^{m\times n}$ and $z_i \in \mathbf{R}^m$ are arbitrary random variables.

7. A numerical example

The theoretical results obtained show that the solvability of GDRE (6) is equivalent to the existence of optimal solution to the LQ problem (1)–(2). Moreover, based on GDRE (6), we can obtain an optimal control with limited calculation. The following numerical example illustrates the procedure of finding the optimal solution.

Consider a three-stage system (1)–(2) with initial state

$$x_1 = \begin{pmatrix} 0.4692 \\ -0.2591 \end{pmatrix}.$$

The coefficients of the dynamics are as follows

$$A_1 = \begin{pmatrix} 0.9501 & -0.6068 \\ 0.2311 & 0.4860 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.8913 & -0.4565 \\ 0.7621 & 0.0185 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0.8214 & -0.6154 \\ 0.4447 & 0.7919 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0.6979 \\ 0.3784 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.8600 \\ 0.8537 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0.5936 \\ 0.4966 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} -0.5681 & 0.7027 \\ 0.3704 & 0.5466 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -0.4449 & 0.6213 \\ 0.6946 & 0.7948 \end{pmatrix},$$
$$C_3 = \begin{pmatrix} -0.9568 & 0.8801 \\ 0.5226 & 0.1730 \end{pmatrix},$$
$$D_1 = \begin{pmatrix} 0.8998 \\ -0.8216 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.6449 \\ -0.8180 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0.6602 \\ -0.3420 \end{pmatrix}.$$

The parameters on the random factors are

$$\rho_1^{xu} = -0.2742, \quad \rho_2^{xu} = 0.5690, \quad \rho_3^{xu} = 0.5803,$$

$$V_1 = \begin{pmatrix} 0.9883 & 0.5031 \\ 0.5031 & 0.5155 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0.3340 & -0.3294 \\ -0.3294 & 0.5798 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0.5678 & 0.4267 \\ 0.4267 & 0.6029 \end{pmatrix}.$$

Finally, the state and control weights are the following

$$Q_1 = \begin{pmatrix} -0.5000 & 0 \\ 0 & 0.2000 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -0.6000 & 0 \\ 0 & -0.6000 \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} 0.8000 & 0 \\ 0 & 0.5000 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 1.0000 & 0 \\ 0 & 0.5000 \end{pmatrix},$$
$$R_1 = -0.9797, \quad R_2 = -0.4072, \quad R_3 = -0.2523.$$

Note that in this example, all the control weights R_i are negative, while some of the state weights Q_i are indefinite. We solve the corresponding GDRE of this problem stage by stage and construct the optimal feedback control law K_i. Finally, we can calculate the optimal cost value.

Specifically, for GDRE (6), the terminal condition is $P_4 = Q_4$.

Stage 3:

$G_3 = R_3 + B_3' P_4 B_3 + D_3' P_4 D_3 = 0.7176,$
$G_3^{\dagger} = G_3^{-1} = 1.3935,$
$H_3 = (0.1795, 0.1514),$
$$P_3 = \begin{pmatrix} 2.5808 & -1.1643 \\ -1.1643 & 1.9500 \end{pmatrix}.$$

The optimal feedback control gain is $K_3 = -G_3^{\dagger} H_3 = (-0.2502, -0.2109)$.

Stage 2:

$G_2 = 4.8196,$
$G_2^{\dagger} = G_2^{-1} = 0.2075,$
$H_2 = (0.0085, -0.6830),$
$$P_2 = \begin{pmatrix} 3.1721 & -0.3630 \\ -0.3630 & 0.9395 \end{pmatrix}.$$

The optimal feedback control gain is $K_2 = (-0.0018, 0.1417)$.

Stage 1:

$G_1 = 4.2472,$
$G_1^{\dagger} = G_1^{-1} = 0.2355,$
$H_1 = (2.5990, -1.6532),$
$$P_1 = \begin{pmatrix} 1.9692 & -1.8864 \\ -1.8864 & 2.7290 \end{pmatrix}.$$

The optimal feedback control gain is $K_1 = (-0.6119, 0.3892)$. Finally, the optimal cost value is

$$J(x_1) = x_1' P_1 x_1 + \sum_{i=1}^{3} \mathbf{T}r(V_i P_{i+1}) = 7.9585.$$

8. Conclusion

In this paper we have investigated the discrete-time stochastic indefinite LQ problem in a general setting, allowing the weighting matrices in the cost function to be indefinite. The underlying system is subject to external perturbations which affect multiplicatively and additively the parameters of the model in both the state and the controls. We have introduced a new Riccati-type equation which plays a central role in solving the indefinite LQ problem. At the same time we have introduced an LMI condition which turns out to be necessary and sufficient for the solvability of our the Riccati equation. More precisely, we have shown that the well-posedness, the attainability of the LQ problem, the feasibility of the LMI and the solvability of the Riccati equation are equivalent to each other. Also, we have provided a complete characterization of all optimal controls.

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